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An Exploration of Pre-service Mathematics Teachers' Routine thinking in Solving Problems in Geometry: A Case Study of a University in Ghana

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ABSTRACT

Concerns have been raised that many mathematics classrooms follow a set of distinctive rules in solving mathematical tasks. Despite the impact of teachers' knowledge on instructional design, and the critical role geometry plays in school mathematics curricula, research shows that pre-service mathematics teachers demonstrate more ritualised (procedural) knowledge in solving geometric problems than explorative (conceptual) knowledge. This study, therefore, explored the nature of eight Ghanaian pre-service mathematics teachers' routine thinking in solving questions in geometry. A test consisting of 16 questions was administered to them. Those who solved 12 or more questions correctly were deemed to have performed well and were assigned to Group A, whilst those who solved fewer than 12 questions were deemed not to have performed so well and were assigned to Group B. The study used the interpretivist paradigm following the qualitative approach. The data was gathered through tests and semi-structured interviews. The results showed that the solution strategies of the three pre-service teachers in Group B were more ritualised, as they solved most of the tasks using a set of procedures. This ritualised thinking was evident in the solutions of those in Group B compared to those in Group A, whose routine strategies were more objectified and property-guided. Following these solution approaches, the study concluded that the Group B pre-service teachers demonstrated more ritualised thinking in solving geometric tasks than explorative thinking, which was more evident in the discourses of those in Group A. It was recommended that mathematics pre-service teachers should be equipped with an explorative sense or reasoning in addition to their ritualised or procedural reasoning. Findings will help develop pre-service mathematics teachers' geometric reasoning by designing targeted instructional activities for learning geometry

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INTRODUCTION

Research shows that the quality of instruction learners receive significantly influences their learning gains.[1] To enable learners to attain the expected competencies in learning geometry, teachers must organise learning experiences that are critical to the development of geometric ideas. In the educational

[1] Armin Jentsch and Lena Schlesinger, "Measuring Instructional Quality in Mathematics Education," in *CERME 10*, 2017; Minju Yi et al., "Measuring Pre-Service Elementary Teachers' Geometry Knowledge for Teaching 2-Dimensional Shapes," *Eurasia Journal of Mathematics, Science and Technology Education* 18, no. 8 (July 12, 2022): em2137, https://doi.org/10.29333/ejmste/12220.

system of every nation, teachers are assigned the responsibility to interpret the curriculum for learners. Thus, the implementation of every education system depends on teachers, making their function vital. The teacher has the role of transforming and representing the subject matter knowledge of the curriculum to learners in an understandable way.[2] Jentsch and Schlesinger outline three dimensions of instructional quality in mathematics education, which are: classroom management, personal learning support and cognitive activation.[3]

Even though all are important, what is worth mentioning, as a critical responsibility of mathematics teachers, is the support they provide for individual learners in terms of guidance and constructive feedback, and how teachers use problem-solving tasks to activate and promote the learning process. Thus, in the context of geometry education, in-service teachers and pre-service teachers (PSTs) need to learn how to design and implement teaching strategies necessary for supporting instructional quality in geometric discourse.[4] Learning, understanding and application of geometric knowledge in solving problems emanate from instructional quality, which is a function of the teachers' depth of geometric knowledge and flexibility in thinking about solving geometric tasks.[5] According to Sfard's commognitive framework, thinking is a form of communication and is individualised in nature.[6] She asserts that communicational activities that bring people together are called discourses. Considering mathematics as a discourse, she identifies routines as one of the main elements. Sfard identifies two types of routines, namely rituals (procedural understanding), which are repetitive patterns or procedures that are often regulated by rules, and explorations as routines that produce new endorsed (conceptual understanding) narratives about mathematical objects.

Despite the impact of teachers' knowledge on instructional design, and the critical role geometry plays in school mathematics curricula, research shows that pre-service mathematics teachers demonstrate more ritualised (procedural) knowledge in solving geometric problems than explorative (conceptual) knowledge.[7] This difficulty may limit their ability to foster their students' deep understanding of geometry. Despite Ghanaian students' low performance in mathematics, with the Chief Examiners' report on their difficulties in solving geometry problems, little or no work has focused on pre-service mathematics teachers' nature of geometry knowledge (routine thinking) in the context of ritualised (procedural), explorative (conceptual). The few existing ones have focused on pre-service geometric thinking using van Hiele's identified levels as a lens of study.[8] This study, therefore, sought to conceptualise pre-service mathematics teachers' routine thinking in solving tasks in geometry using Sfard's commognitive framework as a study lens. The goal of the study was to answer the questions:

1. What are pre-service mathematics teachers' solution approaches to geometry tasks about ritualised or explorative routines?
2. What is the geometric sense of reasoning underlying the pre-service teachers' solution approaches?

Findings from this study will inform teacher education programmes to address and equip pre-service mathematics teachers' geometric reasoning by designing targeted instructional activities for learning geometry. Findings will also help bring about improvement in pre-service mathematics teachers' geometry content knowledge development. For this study, the terms ritualised and explorative routines are used synonymously with ritualised and explorative thinking.

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2 Lee S Shulman, "Those Who Understand: Knowledge Growth in Teaching," *Educational Researcher* 15, no. 2 (1986): 4–14.

3 Jentsch and Schlesinger, "Measuring Instructional Quality in Mathematics Education."

4 Salvador Llinares, "Instructional Quality of Mathematics Teaching and Mathematics Teacher Education," *Journal of Mathematics Teacher Education* 24, no. 1 (February 2021): 1–3, https://doi.org/10.1007/s10857-021-09488-2.

5 Yi et al., "Measuring Pre-Service Elementary Teachers' Geometry Knowledge for Teaching 2-Dimensional Shapes."

6 A. Sfard, *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing* (New York, NY: Cambridge University Press, 2008).

7 K. Sabey, "Secondary Preservice Teachers' Understanding of Euclidean Geometry" (University of Northern Iowa, 2009), https://scholarworks.uni.edu/etd/691.

8 Robert Benjamin Armah, Primrose Otokonor Cofie, and Christopher Adjei Okpoti, "The Geometric Thinking Levels of Pre-Service Teachers in Ghana," *Higher Education Research* 2, no. 3 (2017): 98–106; Ebenezer Bonyah and Ernest Larbi, "Assessing Van Hiele's Geometric Thinking Levels among Elementary Pre-Service Mathematics Teachers.," *African Educational Research Journal* 9, no. 4 (2021): 844–51; Filiz Tuba Dikkartin Övez and Emine Özdemir, "An Examination of Pre-Service Teachers' Van Hiele Levels of Geometric Thinking and Proof Perception Types in Terms of Thinking Processes.," *Educational Research and Reviews* 19, no. 1 (2024): 26–39.

LITERATURE REVIEW

According to Alex and Mammen, Erdogan et al., teachers' proficient geometric thinking should entail an understanding of both the concepts and procedures of the subject matter and how to blend them to foster effective teaching.[9] Sfard supports this view when she explains that understanding both concepts and procedures is equally important in the teaching and learning of mathematics, even though conceptual understanding, which shares the characteristics of explorative routines in Sfard's framework, is the focus of school mathematics.[10] Rittle-Johnson and Schneider consider conceptual understanding to be bidirectional; hence, teachers need to demonstrate an adequate understanding of both for teaching.[11]

Even though one may ask which one comes first, the National Council of Teachers of Mathematics (NCTM) claims that procedural knowledge should not be taught in the absence of conceptual understanding.[12] Also, Sabey remarks that a deep conceptual understanding of the subject matter can prevent learners from using incorrect procedures.[13] Thus, even though both conceptual and procedural understanding are important for mathematical proficiency, conceptual understanding seems to be more valuable for teaching since it enables one to explain the rationale behind rules and procedures.[14] According to Alex and Mammen, mathematics educators pay special attention to developing learners' conceptual understanding because it forms one of the aims of teaching mathematics and is a significant approach to preparing learners for the 21st century.[15]

Conceptual knowledge is an understanding of mathematical concepts that enables one to make connections between them and engage in meaningful learning.[16] Knowing 'how', 'when', and 'why' mathematical concepts apply in various contexts is associated with conceptual understanding. Learners with conceptual understanding learn mathematics by connecting new ideas to what they already possess in their existing schema and also relating these to new situations.[17] A conceptual understanding of mathematics helps students become more flexible and develop their ability to think creatively.[18]

Procedural knowledge, on the other hand, refers to knowledge of rules and procedures. It deals with how to follow rules and routine procedures to solve mathematical tasks and the skills necessary to perform them, accurately, efficiently and with flexibility. It is characterised by the ability to follow and use step-by-step procedures, also known as algorithms, to solve a task.[19] A learner who relies on algorithms can solve a given task but may have little or no understanding of why a certain procedure works in other situations.

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[9] Jogymol Alex and Kuttikkattu J Mammen, "Students' Understanding of Geometry Terminology through the Lens of Van Hiele Theory," *Pythagoras* 39, no. 1 (2018): 1–8; Erdoğan Usta and Çiğdem Akkanat, "Investigating Scientific Creativity Level of Seventh Grade Students," *Procedia - Social and Behavioral Sciences* 191 (June 2015): 1408–15, https://doi.org/10.1016/j.sbspro.2015.04.643; H. E. Zuya, D. B. Matawal, and K. S. Kwalat, " Conceptual and Procedural Knowledge of Pre-Service Teachers in Geometry," *International Journal of Innovative Education Research* 5, no. 1 (2017): 30–38.

[10] Sfard, *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*.

[11] Bethany Rittle-Johnson and Michael Schneider, "Developing Conceptual and Procedural Knowledge of Mathematics," in *The Oxford Handbook of Numerical Cognition* (Oxford University Press, 2014), 1118–34, https://doi.org/10.1093/oxfordhb/9780199642342.013.014.

[12] À NCTM, "National Council of Teachers of Mathematics/2000/Principles and Standards for School Mathematics," *Council of Teachers of Mathematics*, n.d.

[13] Sabey, "Secondary Preservice Teachers' Understanding of Euclidean Geometry."

[14] Lindsay Keazer and Jennifer Phaiah, "Analyzing Prospective Elementary Teachers' Evidence of Conceptual Understanding and Procedural Fluency," *Investigations in Mathematics Learning* 15, no. 2 (April 3, 2023): 135–48, https://doi.org/10.1080/19477503.2022.2139112; Rittle-Johnson and Schneider, "Developing Conceptual and Procedural Knowledge of Mathematics."

[15] Alex and Mammen, "Students' Understanding of Geometry Terminology through the Lens of Van Hiele Theory."

[16] Rittle-Johnson and Schneider, "Developing Conceptual and Procedural Knowledge of Mathematics."

[17] Ahmet Erdogan, "Examining Pre-Service Mathematics Teachers' Conceptual Structures about" Geometry".," *Online Submission* 8, no. 27 (2017): 65–74; D Salim Nahdi and M Gilar Jatisunda, "Conceptual Understanding and Procedural Knowledge: A Case Study on Learning Mathematics of Fractional Material in Elementary School," in *Journal of Physics: Conference Series*, vol. 1477 (IOP Publishing, 2020), 042037; Sabey, "Secondary Preservice Teachers' Understanding of Euclidean Geometry."

[18] Nava Kivkovich, "A Tool for Solving Geometric Problems Using Mediated Mathematical Discourse (for Teachers and Pupils)," *Procedia-Social and Behavioral Sciences* 209 (2015): 519–25; Pedro Antonio Gutierrez et al., "Ordinal Regression Methods: Survey and Experimental Study," *IEEE Transactions on Knowledge and Data Engineering* 28, no. 1 (January 1, 2016): 127–46, https://doi.org/10.1109/TKDE.2015.2457911.

[19] Derek Hurrell, "Conceptual Knowledge OR Procedural Knowledge or Conceptual Knowledge AND Procedural Knowledge: Why the Conjunction Is Important to Teachers," *Australian Journal of Teacher Education* 46, no. 2 (February 2021): 57–71, https://doi.org/10.14221/ajte.2021v46n2.4; Bethany Rittle-Johnson, "Developing Mathematics Knowledge," *Child Development Perspectives* 11, no. 3 (September 4, 2017): 184–90, https://doi.org/10.1111/cdep.12229.

According to Erdogan, incorporating conceptual and procedural knowledge into instruction enables learners to develop and demonstrate high-order mathematical thinking skills.[20] Whilst these two kinds of knowledge are important for teaching, research shows that in-service teachers and pre-service teachers seem to demonstrate more procedural knowledge than conceptual knowledge. For example, Sabey studied 15 pre-service teachers' understanding of Euclidean geometry.[21] The PSTs were taking secondary mathematics education as their study programme at a university. The study used a mixed-methods approach and collected data using both paper-and-pencil tests and interviews. The authors ranked the PSTs by their performance and selected the first three who answered all 15 questions correctly (high-performance) and the three PSTs who correctly answered only eight (about half) of the 15 questions as the average group, whilst those who answered fewer than 8 questions were considered as a low-performing group. Two from the high-performing group, two from the low group, and one from the average group were invited for follow-up interviews. Results showed that the PSTs in the high-performing group demonstrated conceptual and procedural knowledge, whilst those in the low and average group showed deficiencies in knowledge of answering the questions, including conceptual and procedural. The author claimed that there was evidence of most of the PSTs' use of procedural knowledge in their solution strategies. According to Sfard, students stick to procedures and rules because they consider them the easiest way to solve mathematical tasks.[22] Also, classroom instruction mostly focuses on rules rather than teaching for understanding.[23]

Literature shows that the use of a set of rules in solving mathematics tasks is common among students because they think it leads to the expected results.[24] The authors found in the study that many learners resorted to a rule-based approach in solving mathematics questions. This rule-based approach often hinders learners' problem-solving abilities.[25] Akhter and Akhter and Al-Mutawah et al. add that rule-based approach to learning affects learners' conceptual understanding and limits their ability to apply the knowledge in new situations.[26] This limited understanding is often carried to higher levels of learning. According to Sabey, several studies show that most PSTs can solve mathematics tasks by following procedures, but demonstrate a weak conceptual understanding of their solutions.[27] Teachers can teach well when they can demonstrate adequate mathematical thinking and have good pedagogical strategies. This will enable them to develop their learners' understanding of the mathematical concepts they will teach.[28]

THEORETICAL FRAMEWORK

Commognition stems from the theory that a person engages in self-communication by thinking when engaged in an activity. The term 'commognition' is derived from two keywords, viz., cognition (thinking) and communication. According to Sfard, thinking is a form of communication and is individualised in nature.[29] This means that one communicates with oneself whilst thinking. Sfard identifies four constructs for gaining insight into one's thinking: word use, visual mediator, routines and narratives.[30] For the purpose of this paper, only the routine will be explained further. Sfard defines routines as "a set of meta-

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[20] Erdogan, "Examining Pre-Service Mathematics Teachers' Conceptual Structures about" Geometry"."
[21] Sabey, "Secondary Preservice Teachers' Understanding of Euclidean Geometry."
[22] Sfard, *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*.
[23] Sfard, *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*.
[24] Manveer Mann and Mary C Enderson, "Give Me a Formula Not the Concept! Student Preference to Mathematical Problem Solving," *Journal for Advancement of Marketing Education* 25, no. Special Issue on Teaching Innovations in Retailing Education (2017).
[25] Hurrell, "Conceptual Knowledge OR Procedural Knowledge or Conceptual Knowledge AND Procedural Knowledge: Why the Conjunction Is Important to Teachers"; Mann and Enderson, "Give Me a Formula Not the Concept! Student Preference to Mathematical Problem Solving."
[26] Nasrin Akhter and Nasreen Akhter, "Learning in Mathematics: Difficulties and Perceptions of Students.," *Journal of Educational Research (1027-9776)* 21, no. 1 (2018); Masooma Ali Al-Mutawah et al., "Conceptual Understanding, Procedural Knowledge and Problem-Solving Skills in Mathematics: High School Graduates Work Analysis and Standpoints.," *International Journal of Education and Practice* 7, no. 3 (2019): 258–73.
[27] Sabey, "Secondary Preservice Teachers' Understanding of Euclidean Geometry."
[28] I L K Dewi and S Asnawati, "Relationship among Pre-Service Teacher Conception of Geometry and Teaching Skills," in *Journal of Physics: Conference Series*, vol. 1280 (IOP Publishing, 2019), 042042; S. Sugeng and M. C. Nurhanurawati, " The Effect of Various Media Scaffolding on Increasing Understanding of Students' Geometry Concepts," *Journal on Mathematics Education* 9, no. 1 (2018): 95–102.
[29] Sfard, *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*.
[30] Sfard, *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*.

rules that describe a repetitive discursive action". She distinguishes between the 'how' as a procedure-driven solution and 'when' of a routine as those that show understanding and applicability of knowledge.[31] Thus, routines can be ritual or explorative, which is the 'how' or the 'when' respectively.

Sfard asserts that rituals are the distinctive rules that guide actions and are pre-determined by people in authority, such as authors, teachers or lead discussants. Rituals are basic units of discourse and deal with 'how' to get something done.[32] It can be understood as the process of an algorithm that uses a step-by-step approach to perform an activity. In other words, rituals are limited to the 'how' of routines. A person working within rituals can demonstrate high knowledge of procedures but may be limited in knowing or understanding the 'when' or the 'why' underlying such action. Ritualised routines are prone to imitating the knowledgeable person (teacher) or colleague in a discourse. They are considered a basis for transformation to an explorative routine.

Explorative routines are patterned activities that aim to produce or verify an endorsed narrative. Learners who embark on explorations can explain 'when' to use a routine and 'why' the routine works. Exploration is the implicit or explicit understanding of geometric objects of study. Learner development of explorative routines enables him/her to apply knowledge in several situations. Learners would exhibit meta-thinking and are capable of devising multiple approaches to a task, and can also justify their routines.[33] Explorative routines have a great range of applicability to learning mathematical objects. Learners with an explorative sense of reasoning are able to devise problem-guided solutions to mathematical tasks. Their routines are informed by the most efficient properties that are associated with mathematical tasks. Whilst rituals deal with the rules that determine the course of action, which, in most cases, place constraints on the learners' thinking/reasoning, explorations enable diversity of thinking, leading to a demonstration of high-order thinking in learning mathematics.

Learners need to develop fundamental concepts and accurate procedures of routines in mathematical discourse, but these must serve as a basis to develop explorative competencies. Rituals are structured and often require some form of acceptance or approval from the lead discussant. Such discourse, due to its limitation on learners' diversity in reasoning, often affects their range of applicability.[34] Learners develop autonomy in learning when there is phenomenal growth in moving from ritual to explorative routines.[35] Learners who have developed an exploratory routine in a discourse have the capacity to produce endorsed narratives about learned geometric objects. Learners' abilities to demonstrate such endorsed narratives (meaningful ideas) result from being able to make connections across a range of routines. This enables learners to develop a high sense of flexibility about geometric objects and the ability to substantiate such objects.[36]

METHODOLOGY

This study was an interpretive case study that used a qualitative approach. The qualitative case study approach was utilised to gain an understanding of the intricacies of the PSTs' experiences with geometric discourse.

Eight second-year Ghanaian PSTs were selected for the study. The selection was based on their performance on the written test, which gave an initial insight into how the geometric tasks were solved. The test consisted of 16 items based on the senior high school geometry content in the mathematics curriculum. Four of those who solved 12 or more items correctly were deemed to have performed well and placed in Group A (and named Clement, Jones, Maxwell and Stephen), whilst four of those who solved fewer than 12 items were placed in Group B (and named Albert, Alex, Cynthia and Nsiah). Their written solution to the tasks made it possible to explore their sense of reasoning underlying how the solutions were devised with regard to characteristics of ritualised or explorative routines.

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[31] Sfard, *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*. P. 208
[32] Sfard, *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*.
[33] Sfard, *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*.
[34] Regina Miriam Essack, *Exploring Grade 11 Learner Routines on Function from a Commognitive Perspective* (University of the Witwatersrand, Johannesburg (South Africa), 2015).
[35] Sfard, *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*.
[36] Sfard, *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*.

To understand their thinking processes better, they were engaged in semi-structured interviews. This type of interview was used since it allows for follow-up questions to gain in-depth insight into the phenomenon under study.[37] During the interview, the PSTs were asked to explain their solution strategies and why they chose to use such approaches to better understand their line of thought. Best and Khan assert that interviews enable researchers to learn more about participants in the research phenomenon.[38]

The data obtained was analysed using the characteristics of the ritualised and explorative routine as explained in the theoretical framework. Thus, the data was deductively analysed in search of the presence of the two constructs (ritualised and explorative routine) in their discourse.

The department head was consulted for approval before this study could be carried out. The PSTs were invited to participate voluntarily after being informed of the study's objectives and the methods used to generate the data. Since pseudonyms were going to be used in the study, they were promised that whatever data they gathered would be kept secure and anonymous. Thus, people who voluntarily consented to take part in the study were considered participants.

PRESENTATION OF RESULTS

The results presented in this section focused on only the correct solutions the PSTs provided for the tasks. Analysis of these solution strategies was conducted in search of the presence of the characteristics of ritualised and explorative routines as explained in the literature.

Ritualised routine: The use of a step-by-step solution strategy

Task 1 of the written test required the participants to find the value of the unknown angle represented by the letter *m*. Generally, the value could be found by using the angle property on a straight line, or the application of vertically opposite angles, depending on one's understanding. All eight participants solved these tasks correctly using different approaches. Of the eight, four, Maxwell (A), Albert, Nsiah and Alex, all in B, used the angle property on a straight line to solve the task on a step-by-step basis, whilst the remaining four used the emerging vertically opposite angle property. Those who used the straight-line algorithm first calculated the angle, labelled *p* (Nsiah's plan), which is adjacent to 130^0 on a straight line *CD*, before considering the angle *p* together with angles *m* and 60^0 on another straight line. This process is seen in Nsiah's (B) solution to task 1 as shown in Figure 1.

1. Find the size of the angle marked by *m*.

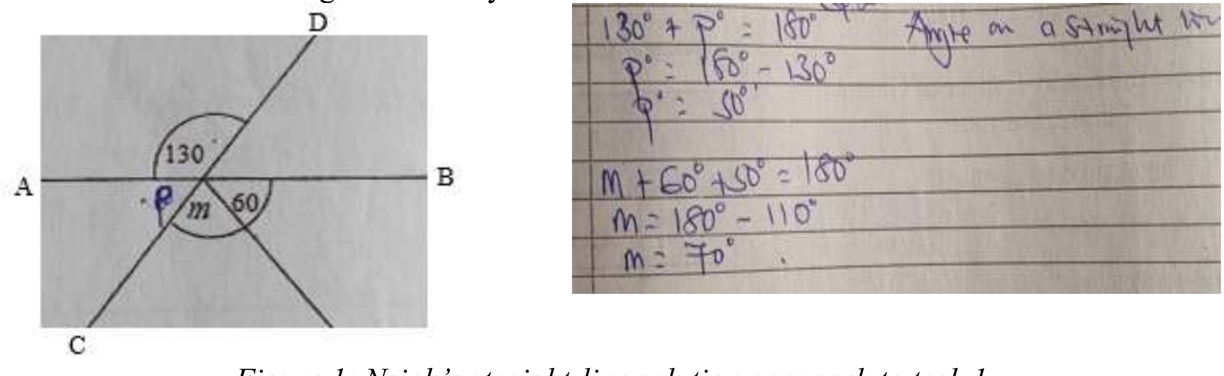


Figure 1: Nsiah's straight-line solution approach to task 1

In trying to understand their thought processes governing the solutions provided, the participants were interviewed as follows:

Researcher: Could you help me understand how you solved for the value of *m*?

Nsiah (B): … *I used angles on a straight line to first find the value of p. With this, I added p and 130^o and equated to 180^o, and subtracting 130^o from 180^o, I got the p to be 50^o. Now, using the*

[37] Melissa DeJonckheere and Lisa M Vaughn, "Semistructured Interviewing in Primary Care Research: A Balance of Relationship and Rigour," *Family Medicine and Community Health* 7, no. 2 (March 8, 2019): e000057, https://doi.org/10.1136/fmch-2018-000057.

[38] Ramin Khaksar et al., "Unmasking Seafood Mislabeling in US Markets: DNA Barcoding as a Unique Technology for Food Authentication and Quality Control," *Food Control* 56 (2015): 71–76.

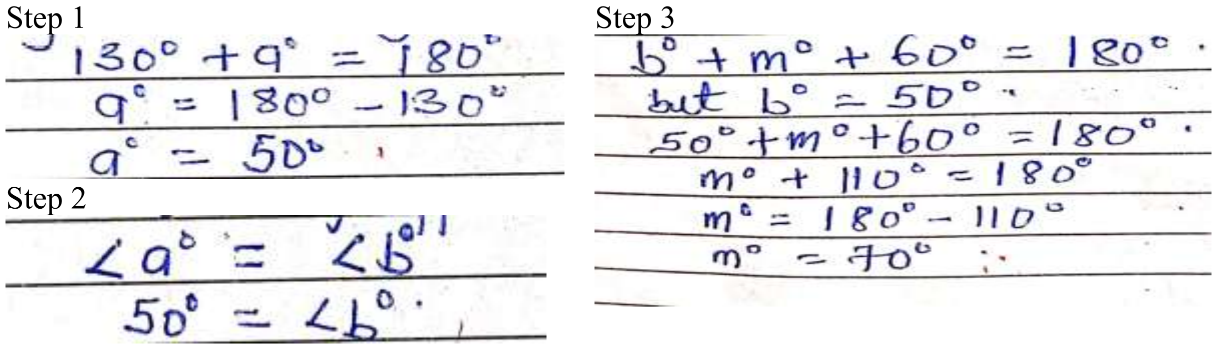
horizontal straight line, I added the three angles and equated them to 180º. I then solved for the value of 'm' which is 70º.

In a similar manner, Albert (B) explained his solution as follows.

… I used the approach of an angle on a straight line. So, I have 130º plus the 'a' so their sum is equal to 180º. So, I calculated for a. Equally, … I have my angles a and b, and I can say they are vertically opposite angles and are equal. … considering this straight line [referring to straight line *AB*] *'b' plus 'm' plus 60º gives me 180º. I make 'm' the subject and solved it to get 70º.*

It is evident from the preceding excerpts, which are similar to the explanations of the other two, that they used the step-by-step approach to work with different straight lines (or the same) to obtain their answers. The participant's ability to devise solutions to the task could result from several processes. These participants, upon visualising the task, may have recognised a straight line which they internalised, to produce the preferred solution. They seemed to have their priori knowledge more linked to straight lines.

Albert's explanation shows that he knew of vertically opposite angles, but he chose to work on the observed straight line. His knowledge of vertically opposite angles is shown in his solution below:



He substantiated the routines in the various steps with the following narratives.

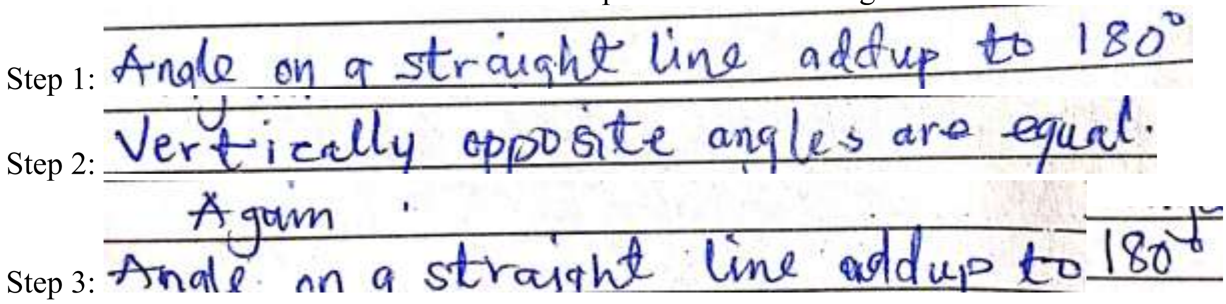


Figure 2: Albert's solution to task 1 with substantiating narratives

He explained that *"I have my angles a and b, and I can say they are vertically opposite angles and are equal"* [Refer to Figure 1, Task 1]. He labelled the angle adjacent to 130º on the straight line *AB* as *a.* He then related the angle value of *a* (50º) to that of *b* (the angle space adjacent to [*m* and 60º]) also on straight line *AB* and supported by the narrative 'vertically opposite angles' (Step 2, in Fig. 2). He used the straight line *AB* to solve for the value of *m*. It is evident from Albert's discourse that he preferred using the straight line in a horizontal appearance or prototype view. Albert's mention of vertically opposite angles, associated with the intersection of the two straight lines *AB* and *CD,* could also mean that he did not exercise his visual ability to see beyond the ordinary, that the angle 130º is also vertically opposite to the sum of (*m* and 60º) and hence, equal in value. Thus, the task was mentally manipulated at a shallow level by Albert.

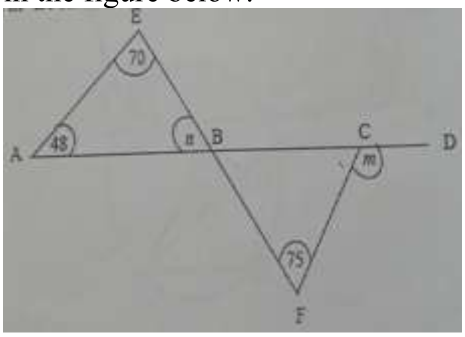
What was different from Nsiah's solution was that his conception of a straight line seemed to be more developed compared to that of Albert. When Nsiah first calculated the adjacent angle to 130º on the straight line *CD*, he then used the value obtained (50º) on the straight line *AB* to calculate the value of *m*, unlike the others (Albert and Alex, B, and Maxwell, A) whose solution showed their preference to work with the straight line AB.

In all the solutions devised using the step-by-step approach, four participants (Maxwell in Group A, and three in Group B, Nsiah, Albert and Alex) showed adequate knowledge of the procedures needed to solve the task and explained their thinking, which showed fluency with the procedure. The participants' ability to solve the task resulted from an internalisation process that took place upon recognition of the straight lines. Internally, they may have connected the straight line seen to what they are familiar with, which possibly informed their thinking, and formulated the equation to match the straight line algorithm.

Some of the participants showed proficient knowledge in using the straight line algorithm and other fundamental concepts in geometry to find answers to the task on triangles. For example, tasks on triangles were designed to gain insight into the participants' preferred approach of using either the 'exterior angle theorem' or the 'straight line algorithm' to solve related questions.

In task 2, both the emerging exterior angle and the straight-line algorithm could be used. It was found that three of the four Group B participants (Albert, Cynthia and Nsiah) calculated the adjacent angle to the exterior angle so that they could easily apply the straight line algorithm to find the correct answer. They all explained their solutions with a clear sense of knowledge and substantiated each step with an appropriate geometry property. Cynthia's (B) solution (typed for legibility) is shown in Figure 3.

Task 2. Find the values of *n* and *m* in the figure below.



$70^o + 48^o + n$
$= 180^o$ $(Angle\ in\ triangle\ sum\ up\ to\ 180)$
$118^o + n = 180$
$n = 180^o - 118^o$
$n = 62^o$
ΔBCF:
$a + 75^o + b$
$= 180^o$ $(Angle\ in\ a\ triangle\ sum\ up\ to\ 180$
$but\ a^o = 62^o$ $(a\ is\ direct\ opposite\ angle\ to\ n)$
$62^o + 75^o + b^o = 180^o$
$137^o + b^o = 180^o$
$b^o = 180^o - 137^o$
$b^o = 43^o$
$m + b^o = 180^o$(straight angle)
$m + 43^o = 180^o$
$m = 180^o - 43^o$
$m = 137^o$
$\therefore n = 62^o$ and $m = 137^o$

Figure 3: Cynthia's straight-line solution approach to task 2

Cynthia's approach is a representational solution of two others in Group B, who solved for *m* using the straight line approach. All the Group A participants, together with Alex (B), explained that they could add angles 62˚ (calculated value of *n*) and 75˚, as the two interior opposite angles, to get the value of the exterior angle *m as* 137°.

The solution strategies devised by the participants were based on how they thought about the task. For example, when Cynthia was asked why she solved the task that way, she said, "*I used this approach because of the properties I see within this diagram.*" This means that Cynthia's solution was an externalisation of her internalisation process about the task. Her word use of *"properties I see within the diagram"* could mean that her action of solving the task was connected to her visualisation process that stimulated her mental action. Thus, there was a relation between visual senses and mental action in planning to solve the task.

Explorative routine: Solutions guided by associated geometric properties of the tasks
Sfard notes that a key feature of explorative routine is the ability to produce an endorsed narrative about the properties of an object.[39] Results from the study showed that some of the PSTs (from both Groups) focused their attention on the properties associated with the task design. This meant their discourses about the solution strategies were more of justifying with the objectified properties that necessitated the preferred routine. This solution strategy was mostly found in the discourses of the PSTs in Group A as compared to those from Group B. The Group A participants substantiated their discursive actions by producing endorsed geometric properties about the task. It was found that the participants' plan for devising solutions was based on identifying the most associated properties of the task. This is evident in the extract of Stephen's (A) response to task 1 as follows:

> *...from the diagram given, I can say that the angle 130^o is vertically opposite to the two angles (m and 60^o). Therefore, knowing that vertically opposite angles are equal, I can write it as ...*

Also, task 3 received property-guided planning in devising solutions, as evidenced in the discourses of some of the participants.

Task 3. Find the values of the angles marked by letters.

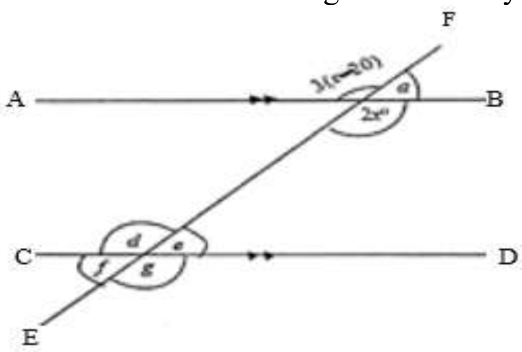


Figure 4: Question on the angle properties of parallel lines

When Jones (A) was engaged in an interview to explain his solution plan, he explained that:

> *... from the diagram, we can see that* $3(x-20)$ *it is vertically opposite to* $2x$ *which are equal. So, I equate the two angles ...*

A similar explanation with a focus on identifying the related properties was found in Cynthia's (B) discourse when she said:

> *... we have direct* [vertically] *opposite angles here which are 3(x – 20) and 2x* [points to the angles] *which are equal....*

Their responses show that they have developed the competence to engage in a critical exploration of the task design and to identify its objectified properties. Thus, their solution strategies were informed by identifying the associated properties of the task. Participants (in both Groups), who engaged in a shallow visual manipulation or exploration of the tasks preferred working with the straight-line algorithm.

The ability to identify the associated properties could be attributed to their critical visual engagement with mental manipulation of the task(s). Mudaly maintains that we manipulate objects in our minds to "create a deeper understanding".[40] This process resulted in their discursive actions, which align with the explorative way of thinking.[41] They were less prompted to justify why they chose a particular

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[39] Sfard, *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*.
[40] Ronicka Mudaly and Sebastian Sanjigadu, "Epistemic Journeying across Abyssal Lines of Thinking: Towards Reclaiming Southern Voices," *Education as Change* 26, no. 1 (2022): 1–29. p.2)
[41] Sfard, *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*.

solution path. Their solution path was probably based on their ability to explore the task further to identify associated properties that could be applied. It is worth mentioning that Alex, Albert and Nsiah, all in Group B, and Maxwell in Group A, provided a series of solutions

by first solving for the value of *a,* the adjacent angle to $3(x-20)$, before finding the value of x. Thus, they could not readily identify that $3(x-20)$ and $2x$ were vertically opposite angles that could be solved straightforwardly by equating them.

The Group A participants produced similar objectified properties about their plans to devise solutions to tasks on triangles. The tasks on triangles were designed so that the exterior angle would draw the PSTs' attention to the exterior angle theorem of a triangle. Explorative thinking was evident in the explanations of Group A participants about their solution to triangles, compared to their counterparts in Group B. For example, in task 2 (see Figure 3), all the participants in Group A, together with Alex in Group B, recognised the angle to be found as an exterior angle and hence, used the exterior angle theorem in their solution to the task as compared with others who used series of solutions, by applying the straight line algorithm before finding the exterior angle (See Cynthia's solution in Figure 3). Some of them gave an explanation that indicated their understanding of how the theorem was developed, using the 'sum of angles in a triangle' and 'the adjacent angles on a straight line'. Hence, they felt confident in applying the theorem to the task. The following extracts depict their explanations.

> *... this is an exterior angle* [pointing to *m*] … [the sum of the] *two ... opposite* [interior] *angles will be equal to the exterior angle. ... you know 62^o we add to 75^o and you get 137^o that will be equal to the exterior angle ... m* (Alex, B).
>
> *… I should find ... m. For m, ... is an exterior angle… the sum of two opposite interior angles of the triangle* [pointing to angle 75^o and angle CBF]*, labeled n. So, this n and 75^o, when you sum them, it is going to give me m ... the exterior angle. So, then I can say the m is equal to 75^o plus 62^o. Then adding, m is 137°* (Jones, A).
>
> *... having found the two opposite interior angles for my m, ... m my exterior angle is equal to 75^o plus 62^o which gave me 137^o* (Stephen, A).

Similar property-guided thinking was evident in Jones' (A) discourse on task 4 as shown in Figure 5.

Task 4: In the figure, PQRS and PUT are straight lines and $\angle PQU = 120^o$. If $|PQ| = |QU|$, find $\angle URS$.

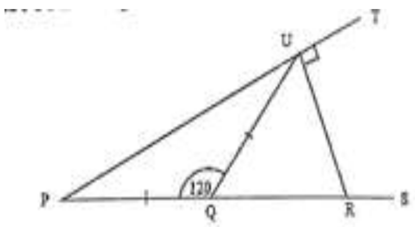


Figure 5: A question assessing the PSTs' knowledge about the properties of triangles

His discursive actions, as seen in the extract below, were based on identifying properties associated with the task. He may have frequently engaged with geometric tasks and understood that the easiest way of solving such tasks is by applying associated properties. Three of the participants in Group A (Stephen, Jones and Clement) identified the angle as an exterior angle and solved using the exterior angle theorem. The rest of the participants' responses focused on the use of angle properties in a triangle and the straight line algorithm. Jones' (A) response is shown below.

> Researcher: Can you explain how you organised your thinking to solve the task?
>
> Jones: *...to find or calculate for angle URS ... the angle here* [moving the pen along the arms *UR* and *RS*]. *I planned to use the exterior angle theorem because of its position. So that if I get*

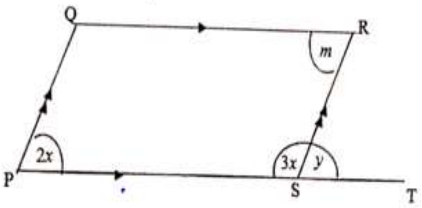
> *here* [pointing to angle *QUR*] *and this angle* [pointing at angle *UQR*], *then I sum them and use the exterior angle theorem.*

The discourses of the three Group A participants were a demonstration of explorative thinking due to their ability to produce associated properties to the task. The rest probably used the straight-line algorithm, possibly because they felt their answers would be guaranteed through the rule-based discourse.

A similar explorative routine was evident in some of the participants' solutions for tasks on the parallelogram. The characteristic feature of the explorative routine is the application of the emerging properties of the task. This routine, guided by the properties, led them to use fewer steps in solving for the angled variables (efficient solution). The participants who produced an explorative routine showed that they were capable of engaging in mathematical thinking by producing objectified properties as a guide to solving geometric tasks.[42] This was evident in their word use and their attention to some visual cues in solving the task. It was evident in the planning phase that they focused attention on the position of the angle to be found. Participants were often found to be moving their pens around the angles and the lines, which is an indication of the utility of their visual perspectives as their major element of reasoning.

An explorative routine is characterised by the ability to apply the concepts of an object in a practical solution to its related task. In this case, the learner is said to have developed the competence to analyse, discern and solve problems related to the learned task.[43] According to Hurrell, explorative thinking is the ability to link the knowledge gained to other mathematical objects.[44] This competence was seen in the routines of the Group A participants and two from Group B.

Task 5. Find the value of the angles marked by letters in the figure below



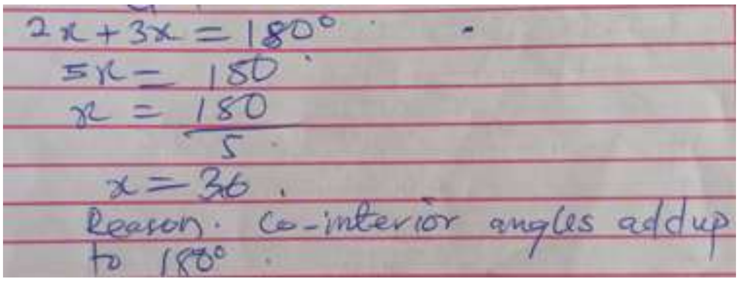


Figure 6: Albert's objectified solution to task 5

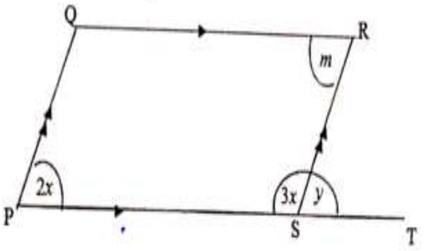
Stephen (A), Albert (B) and Cynthia (B), when engaged with task 5 on the parallelogram in Figure 6, analysed it based on the iconic mediators on the line segments. Their approach may have been enhanced by their internalisation processes. As a result, they each formulated the equation $2x+3x=180^o$ and substantiated it as the co-interior angles. Maxwell and Jones in (A) also solved the task by applying the objectified properties of a parallelogram, using the equality of opposite angles and the sum of interior angles of the generic shape [quadrilaterals], which is 360°. He formulated the equation $2x+3x+2x+3x=360^o$ and solved by demonstrating proficient algebraic skills when explaining what he did. Clement (A) made some mathematical arguments, informed by the observed properties, to arrive at the same initial strategy, devised by Stephen (A), Albert and Cynthia in Group B. These arguments were based on the properties of corresponding angles and the angle sum on a straight line. This is shown in his solution below:

[42] Sfard, *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*.

[43] Hurrell, "Conceptual Knowledge OR Procedural Knowledge or Conceptual Knowledge AND Procedural Knowledge: Why the Conjunction Is Important to Teachers"; Sfard, *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*.

[44] Hurrell, "Conceptual Knowledge OR Procedural Knowledge or Conceptual Knowledge AND Procedural Knowledge: Why the Conjunction Is Important to Teachers."

Task 5. Find the value of the angles marked by letters in the figure below



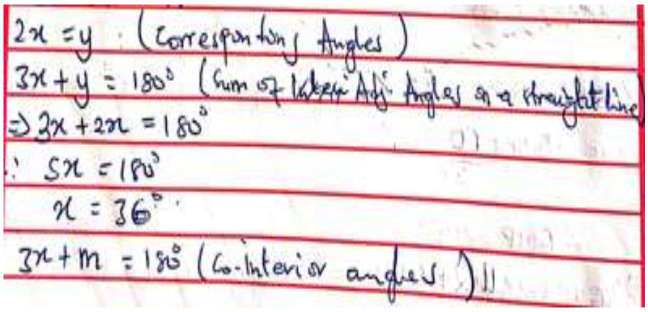


Figure 7: Clement's solution through critical reasoning

Clement's (A) solution shows evidence of engaging in critical thinking to devise a possible solution. Not being able to readily identify that $2x$ and $3x$ were co-interior angles, he resorted to reasoning by making mathematical arguments to arrive at the same initial equation that Stephen (A), Albert (B) and Cynthia (B) started with. Table 1 shows the summary of the PSTs' performance on the tasks with regard to the characteristics of ritualised and explorative routines.

Table 1: Summary of the PSTs' Performances in line with ritual or explorative routine

	Ritual routine	**Explorative routine**
Task 1	Maxwell (A) and Albert, Alex and Nsiah all in (B)	Stephen, Jones and Clement all in (A) and Cynthia (B)
Task 2	Albert, Nsiah and Cynthia all in (B)	All PSTs in Group A and Alex (B)
Task 3	Maxwell (A) and Albert, Alex, and Nsiah all in (B)	Stephen, Jones and Clement all in (A) and Cynthia (B)
Task 4	Maxwell (A) and all PSTs in Group B	Stephen, Jones and Clement in Group A
Task 5	Alex and Nsiah in Group B	All PSTs in Group A and Albert and Cynthia in Group B

DISCUSSION

The results showed that, comparing the performances of the PSTs in the two groups, almost all the solutions and the responses of three in Group B mostly showed the use of a set of procedures to solve the tasks. Generally, the Group B participants probably wanted to work with a clearer focus. This could be an approach they may be comfortable with and know leads to the correct answer, hence their familiarity with it. Mann and Enderson partly attribute learners' preference for a procedure-driven approach to solving mathematical tasks to their ease of remembering and their familiarity with such an approach.[45] This finding is in line with the literature, which shows that learners often prefer solving tasks using a set of procedures.[46] Zuya et al. claim that most learners rely on the use of algorithms to solve tasks in mathematics because they find it more comfortable and familiar to use.[47] For example, a study conducted by Mann and Enderson to assess learners' preference for a procedure (rule or formula-driven) rather than concept-driven approaches found that learners preferred the use of a rule-based approach (procedure) to the conceptual approach.[48] Placing this within the commognitive framework, the use of a set of procedures to solve a task is characteristic of a ritualised routine.[49] Sfard asserts that ritual routines are characterised

[45] Mann and Enderson, "Give Me a Formula Not the Concept! Student Preference to Mathematical Problem Solving."
[46] Akhter and Akhter, "Learning in Mathematics: Difficulties and Perceptions of Students."; Zuya, Matawal, and Kwalat, " Conceptual and Procedural Knowledge of Pre-Service Teachers in Geometry."
[47] Zuya, Matawal, and Kwalat, " Conceptual and Procedural Knowledge of Pre-Service Teachers in Geometry."
[48] Mann and Enderson, "Give Me a Formula Not the Concept! Student Preference to Mathematical Problem Solving."
[49] Sfard, *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*.

by following strict rules, mostly determined by the teacher or authority.[50] It is concerned with 'how' to get something done, with no focus on 'when' or 'why' the approach works.

Group B's use of basic geometric properties is in line with Sfard's assertion that learning a new concept begins with imitating an authority by following a set of procedures.[51] Among the basic concepts of teaching plane geometry are 'the sum of adjacent angles on a straight line' and others, such as the 'sum of interior angles in a triangle', for learning polygons. These concepts serve as a foundation for learning geometry. For example, these two basic concepts are used to prove the exterior angle theorem of a triangle, for which learners (and teachers) are supposed to demonstrate implicit and explicit understanding, and apply them to solve related tasks. Thus, both fundamental thinking and high-concept thinking are necessary for holistic understanding and use in solving problems.[52] Zuya et al. uphold that knowledge of both fundamental and enhanced concepts in geometry and other topics in mathematics is needed for teachers to be effective in teaching.[53] Sfard advises that, as learners are taught procedures as a way of imitating the knowledgeable other (teacher), the procedures must also serve as a basis to transition to explorative thinking.[54] Even though both ritual and explorative routines are good for learning, Sfard maintains that school mathematics aims at producing learners with explorative ways of reasoning.[55] Ritualised (procedure-driven) solutions were discussed in the literature, where research shows that most classroom instructions often focus on how to solve problems by showing learners methods and algorithms, as a way of creating familiarity with such questions.[56] This practice sometimes enables learners to memorise such procedures, with little or no understanding of the underlying concepts. This practice often results in difficulty in using knowledge in new situations.

Among the discourses of the three Group A participants (Clement, Jones and Stephen) were exploratory thinking of producing objectified properties of the task design. The demonstration of such thinking means that they have a good understanding and the mental ability to interpret information in a meaningful way, as noted by Nahdi & Jatisunda.[57] In an explorative routine, discursive actions are based on understanding the mathematical principles underlying one's routine and the ability to explain and substantiate why such an action is taken.[58] Such discourse is the "product of a process that connects prior knowledge with new knowledge".[59] The Group A participants' competence in engaging in explorative routine demonstrates that they have developed rich connections between the properties of geometric shapes used in the study. They used the exterior angle in relation to the tasks on the triangle that required such an approach, as the most associated property guided way of devising solutions.

According to Dewi and Asnawati, teachers should show a clear understanding of geometric concepts so that their knowledge of applying these concepts in solving tasks is well-informed.[60] In this study, the PSTs who produced property-guided solutions performed beyond the use of strict rules (straight line algorithm), which was dominant in the Group B participants. In most of the tasks, the Group A participants (Clement, Jones and Stephen) showed proficient knowledge in devising solutions using geometric concepts and properties of geometric figures and shapes. This geometric competence was evident in their solution processes. During the interview, the participants in Group A substantiated their

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[50] A. Sfard, "When the Rules of Discourse Change, but Nobody Tells You: Making Sense of Mathematics Learning from a Commognitive Standpoint," *The Journal of the Learning Science* 16, no. 4 (2007): 567–615; Clive Gray, "Managing Cultural Policy: Pitfalls And Prospects," *Public Administration* 87, no. 3 (September 27, 2009): 574–85, https://doi.org/10.1111/j.1467-9299.2008.01748.x.

[51] Anna Sfard et al., "Moving Between Discourses: From Learning-As-Acquisition To Learning-As-Participation," in *AIP Conference Proceedings*, 2009, 55–58, https://doi.org/10.1063/1.3266753.

[52] Al-Mutawah et al., "Conceptual Understanding, Procedural Knowledge and Problem-Solving Skills in Mathematics: High School Graduates Work Analysis and Standpoints."; Zuya, Matawal, and Kwalat, " Conceptual and Procedural Knowledge of Pre-Service Teachers in Geometry."

[53] Zuya, Matawal, and Kwalat, " Conceptual and Procedural Knowledge of Pre-Service Teachers in Geometry."

[54] Sfard, *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*.

[55] Sfard, *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*.

[56] Sfard, *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*.

[57] Nahdi and Jatisunda, "Conceptual Understanding and Procedural Knowledge: A Case Study on Learning Mathematics of Fractional Material in Elementary School."

[58] Sfard, *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*.

[59] Nahdi and Jatisunda, "Conceptual Understanding and Procedural Knowledge: A Case Study on Learning Mathematics of Fractional Material in Elementary School." p.2

[60] Dewi and Asnawati, "Relationship among Pre-Service Teacher Conception of Geometry and Teaching Skills."

discursive actions with endorsed narratives, which is a feature of explorative routine.[61] These competencies enabled them to devise clear solutions to the tasks. The PSTs' ability to develop such competence in thinking was an indication that they have the potential to plan and work with the associated geometric properties based on the design of the task, to enhance learners' understanding.[62] The ability to discern this property of the task is a demonstration of high, visually informed thinking. Dewi and Asnawati (2019) assert that teachers or learners with good visual abilities can explore and apply the most appropriate strategies to solve mathematical tasks.[63] Some of the discourses of the PSTs showed that they connected their reasoning abilities to their visual abilities. Mudaly asserts that a person's physical action of devising a solution to a task is related to his/her visualisation process.[64] He further claims that "it would be difficult to conceive of a process where no mental action or image has occurred" (p. 2).

It was also found that even though one of the PSTs in Group A could not identify the objectified property associated with the task on quadrilaterals, he made mathematical connections between the properties to arrive at the expected answer. This finding is in line with Dewi and Asnawati's assertion that learners' ability to learn geometry is based on their tendency to make mathematical arguments using geometric concepts and properties.[65] It forms the basis of building learners' critical reasoning abilities.

RECOMMENDATIONS

To enhance pre-service mathematics teachers' geometry thinking fluency, it is recommended that mathematics teacher educators should have a deep focus on developing a sound and deep geometric thinking of the pre-service teachers, with emphasis on an explorative sense of reasoning in addition to their ritualised or procedural reasoning. It will be needful to organise a learning environment that will enable PSTs to explain their solution strategies with support of the underlying geometric properties since they will need this as a backing force for effective teaching of geometry in future.

CONCLUSION

The study explored Ghanaian pre-service mathematics teachers' routine thinking in solving problems in geometry using the commognitive framework routine thinking in terms of ritualised or explorative thinking as a lens of study. The findings showed that the PSTs' routine strategies for devising solutions to the task were mainly ritualised for those in Group B, whilst those in Group A showed routines that were explorative in nature. It was evident from the Group B PSTs' solutions that many used a step-by-step approach to solve for the required angled values represented by the variables in the geometric tasks. Findings also showed that routine strategies of the PSTs in Group A in devising solutions to the tasks were more explorative. In many cases, they started the solution by identifying the emerging geometric properties associated with tasks. Their solutions were more property-guided, which went beyond the use of basic geometric properties. Thus, the PSTs' solution approaches to geometry tasks were both ritualised and explorative, with many ritualised solutions among those who solved fewer tasks than those who solved many of the tasks and vice versa. Also, those who solved more tasks showed a higher sense of geometric reasoning underlying their solutions compared to those who solved fewer tasks.

Data Availability

Data for the research will be available upon request.

Acknowledgment

The researchers would like to thank the participants in this research for their time.

[61] Sfard, *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*.
[62] Dewi and Asnawati, "Relationship among Pre-Service Teacher Conception of Geometry and Teaching Skills"; Sugeng and Nurhanurawati, " The Effect of Various Media Scaffolding on Increasing Understanding of Students' Geometry Concepts."
[63] Dewi and Asnawati, "Relationship among Pre-Service Teacher Conception of Geometry and Teaching Skills."
[64] Mudaly and Sanjigadu, "Epistemic Journeying across Abyssal Lines of Thinking: Towards Reclaiming Southern Voices."
[65] Dewi and Asnawati, "Relationship among Pre-Service Teacher Conception of Geometry and Teaching Skills."

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